Curves and surfaces

Week 4

Last time:

- Curvature vector $K_\gamma(t) = \frac{1}{V_\gamma(t)} \frac{d}{dt} T_\gamma(t)$
  - In natural parametrization

$$K_\gamma(s) = \frac{d}{ds} T_\gamma(s)$$

Note: $K_\gamma \perp T_\gamma$

- Curvature: $\kappa_\gamma(t) = \|K_\gamma(t)\|$

<u>Acceleration formula</u>: If $\gamma: I \to \mathbb{R}^n$ is a regular C^2 cure:

$$\boxed{\ddot{\gamma}(u) = \underbrace{(V_\gamma(u))^2 \cdot K_\gamma(u)}_{\text{normal acceleration}} + \underbrace{\dot{V}_\gamma(u) \cdot T_\gamma(u)}_{\text{Tangential acceleration}}} \quad , \quad \dot{V}_\gamma(u) = \langle \ddot{\gamma}, T_\gamma \rangle \cdot T_\gamma$$

- In natural parametrization:

$$\ddot{\gamma}(s) = K_\gamma(s)$$

Also:

- Equation for circle: $\gamma(t) + R^2 \cdot K_\gamma(t) = p$
  - The above equation characterizes a circle.

    Any curve satisfying the above, when parametrized with the natural parameter yields

$$\gamma(s) + R^2 \cdot \ddot{\gamma}(s) = p$$

If $\gamma(s) = (x(s), y(s))$, the above has unique solution

$$(x(s), y(s)) = (p_1, p_2) + R \cdot \left(\cos\left(\frac{s}{R} + \phi_0\right), \sin\left(\frac{s}{R} + \phi_0\right)\right)$$

Proposition.
A curve $\gamma : [a,b] \to \mathbb{R}^n$ has $\kappa_\gamma \equiv 0$ if and only if it's (up to reparametrization) a straight line segment.

Proof.
Assume that γ is naturally parametrized (otherwise reparametrize it)

Then $\kappa_\gamma = 0 \iff \|K_\gamma\| = 0 \underset{K_\gamma = \ddot{\gamma}}{\overset{\text{natural par.}}{\iff}} \ddot{\gamma}(s) = 0 \quad \forall s \in [a,b]$

$\iff \gamma(s) = s\,v + q \;, \quad v = \dot{\gamma}(a)$

$\blacksquare$

Lemma. Assume that γ is biregular at u, then

$$K_\gamma(u) = \kappa_\gamma(u) \cdot N_\gamma(u), \quad \text{where } N_\gamma : \text{principal normal.}$$

Proof. When γ is biregular, the principal normal was defined by

$$N_\gamma(u) = \frac{\ddot{\gamma}(u) - \langle \ddot{\gamma}(u), T_\gamma(u) \rangle T_\gamma(u)}{\| \ddot{\gamma}(u) - \langle \ddot{\gamma}(u), T_\gamma(u) \rangle T_\gamma(u) \|}$$

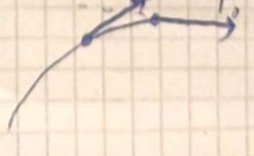
From the acceleration formula: $\ddot{\gamma}(u) - \langle \ddot{\gamma}(u), T_\gamma(u) \rangle T_\gamma(u)$

$$= (v_\gamma(u))^2 \cdot K_\gamma(u)$$

$\blacksquare$

Proposition: Let $\gamma : I \to \mathbb{R}^2$ be C^2 and regular. Then

$\kappa_\gamma(s)$ is equal to the rotation speed of T_γ (with respect to the natural parameter.



__Proof:__ Angle between $T_\gamma(s_0)$ and $T_\gamma(s)$:

$$\theta(s_0,s) = \angle\left(T_\gamma(s_0), T_\gamma(s)\right), \quad \text{satisfies (since } \|T_\gamma\| = 1)$$

$$\|T_\gamma(s_0) - T_\gamma(s)\| = 2\sin\frac{\theta(s_0,s)}{2}$$

So $$k_\gamma(s_0) = \lim_{s\to s_0} \frac{\|T_\gamma(s_0) - T_\gamma(s)\|}{|s-s_0|} = \lim_{s\to s_0} \frac{2\cdot\sin\frac{\theta(s_0,s)}{2}}{|s-s_0|} =$$

$$= \lim_{s\to s_0^+} \frac{2\sin\left(\frac{\theta(s_0,s)}{2}\right)}{s-s_0} \overset{\theta(s_0,s_0)=0}{=\!=} \frac{d}{ds}\left(2\cdot\sin\frac{\theta(s_0,s)}{2}\right)\Bigg|_{s=s_0}$$

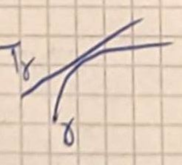
$$= \frac{d}{ds}\theta(s_0,s)\Big|_{s=s_0}$$

Contact order of curves

__Def:__ Let $\alpha, \beta : I \to \mathbb{R}^n$ be two curves of class C^k. We will say that they <u>have contact of order k</u> at t_0 if $\alpha(t_0) = \beta(t_0)$ and

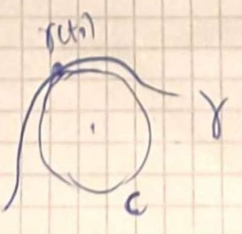
$$\frac{d^m\alpha}{dt^m}(t_0) = \frac{d^m\beta}{dt^m}(t_0) \quad \text{for } 1 \leq m \leq k.$$

- Example: Tangent line: Contact of order 1



- Osculating (= "kissing" in Latin) curves: Contact of order > 1.

- Osculating circle



c: Center $p = \gamma(t_0) + R^2 K_\gamma(t_0)$

$$R = \frac{1}{k_\gamma(t_0)}$$

- If two circles have contact order 2: they are the same circle.

- If two curves have contact of order ≥ 2 at t_0:

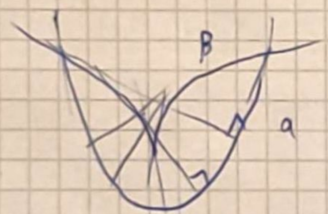
  Same $T_\gamma(t_0)$, $N_\gamma(t_0)$, $K_\gamma(t_0)$

<u>Definition</u>:

If $\alpha : I \to \mathbb{R}^2$ is biregular, the curve of the centers of curvature

$$\beta(t) = \alpha(t) + \underbrace{r_\alpha(t)}_{= \frac{1}{K_\alpha(t)}} N_\alpha(t)$$

is the <u>evolute</u> of α. α is the <u>development</u> of β



Relation: β the curve tangent to the normal lines to α.

α: The curve you obtain if you start unrolling a string from β

Also: For α: The curve β corresponds to the set of points on the plane such that the function $x \to \min_{y \in \alpha} d(x,y)$ ~~dist(x,a)~~ has a "double" point of minimum y on α

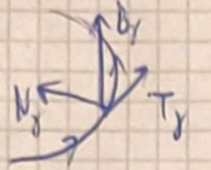
<u>Curves in $\mathbb{R}^3$: The Frenet frame.</u>

Let $\gamma : I \to \mathbb{R}^3$ be a <u>biregular</u> curve, T_γ: unit tangent, N_γ: principal normal

<u>Def</u>: <u>Binormal</u> vector field on γ:

$$B_\gamma(t) = T_\gamma(t) \times N_\gamma(t)$$

Frenet frame of γ: $\{T_\gamma(t), N_\gamma(t), B_\gamma(t)\}$: Orthonormal, positively oriented



The plane spanned by $\{T_\gamma, N_\gamma\}$: osculating plane

" $\{N_\gamma, B_\gamma\}$: normal plane

" $\{T_\gamma, B_\gamma\}$: rectifying plane

Lemma: $B_\gamma = \dfrac{\dot\gamma \times \ddot\gamma}{\|\dot\gamma \times \ddot\gamma\|}$.

Proof: $T_\gamma = \dfrac{\dot\gamma}{\|\dot\gamma\|}$, $N_\gamma = \dfrac{1}{\|\ddot\gamma - \langle \ddot\gamma, T\rangle T\|} \cdot (\ddot\gamma - \langle \ddot\gamma, T\rangle T)$

So $B_\gamma = T_\gamma \times N_\gamma = \lambda \cdot (\dot\gamma \times \ddot\gamma - \langle \ddot\gamma, T\rangle \underbrace{\cancel{\dot\gamma \times T}}_{0})$

$\parallel \dot\gamma \times \ddot\gamma$

and $\|T_\gamma \times N_\gamma\| = 1$ (since $T_\gamma \perp N_\gamma$, $\|T_\gamma\| = 1 = \|N_\gamma\|$)

So $B_\gamma = \dfrac{\dot\gamma \times \ddot\gamma}{\|\dot\gamma \times \ddot\gamma\|}$. $\square$

Definition: A curve $\gamma : I \to \mathbb{R}^3$ is <u>regular in the sense of Frenet</u> if it is biregular and N_γ is a C^1 vector field along γ.

<u>Torsion</u> of a curve which is Frenet regular:

$$\tau(t) = \frac{\langle \dot N(t), B(t)\rangle}{v_\gamma(t)}$$

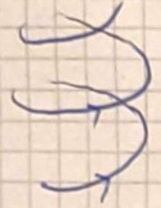
(If γ is C^3 and biregular: it is Frenet regular)

<u>Exercise</u>: The torsion is a <u>geometric quantity</u> (like the curvature)

So we have two geometric scalars associated with a curve in $\mathbb{R}^3$: κ_γ and τ_γ

Example: Circular helix in $\mathbb{R}^3$

$$\gamma(t) = (R\cdot\cos t,\ R\cdot\sin t,\ bt) \quad , R, b > 0$$



Then: $\dot\gamma(t) = (-R\cdot\sin t,\ R\cos t,\ b)$

$\ddot\gamma(t) = (-R\cos t,\ -R\sin t,\ 0)$

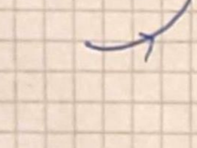
So $(\dot\gamma \times \ddot\gamma \neq 0)$: biregular and smooth $\rightsquigarrow$ It is Frenet regular.

Speed: $v_\gamma(t) = \sqrt{R^2+b^2}$, $T_\gamma = \frac{\dot\gamma}{v_\gamma} = \left(-\frac{R}{\sqrt{R^2+b^2}}\sin t,\ \frac{R}{\sqrt{R^2+b^2}}\cos t,\ \frac{b}{\sqrt{R^2+b^2}}\right)$

~~Curvature vector~~

Principal normal: $N_\gamma = (-\cos t,\ -\sin t,\ 0)$

So $B_\gamma = T_\gamma \times N_\gamma = \left(\frac{b}{\sqrt{R^2+b^2}}\sin t,\ -\frac{b}{\sqrt{R^2+b^2}}\cos t,\ \frac{R}{\sqrt{R^2+b^2}}\right)$



Curvature: $\kappa_\gamma = \left\| \frac{1}{v_\gamma}\frac{d}{dt} T_\gamma \right\| = \frac{R}{R^2+b^2}$

Torsion: $\tau_\gamma = \left\langle \frac{1}{v_\gamma}\frac{d}{dt} N_\gamma, B_\gamma \right\rangle = \frac{b}{R^2+b^2}$

Theorem (Serret-Frenet formulas): Let $\gamma: I \to \mathbb{R}^3$ be Frenet-regular. Then the vector fields $\{T_\gamma, N_\gamma, B_\gamma\}$ are of class C^1 and

$$\frac{1}{v_\gamma}\dot T_\gamma = \kappa_\gamma \cdot N_\gamma$$
$$\frac{1}{v_\gamma}\dot N_\gamma = -\kappa_\gamma\cdot T_\gamma + \tau_\gamma \cdot B_\gamma$$
$$\frac{1}{v_\gamma}\cdot \dot B_\gamma = -\tau_\gamma \cdot N_\gamma$$

or

$$\frac{1}{v_\gamma}\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Proof: Since γ is Frenet regular $\Rightarrow$ γ is C^2 and N is C^1

Since γ is C^2 $\Rightarrow$ $\dot{\gamma}$ is C^1 $\xRightarrow{\dot{\gamma}\neq 0}$ T is C^1

So $B = T \times N$ is also C^1

And: $\frac{1}{v}\dot{T} \overset{\text{definition of } K}{=\!=} K = \kappa \cdot N$ ①

We have: $\kappa = \langle \frac{1}{v}\dot{T}, N \rangle$ ⓐ

Definition of torsion: $\tau = \langle \frac{1}{v}\dot{N}, B \rangle$ ⓑ

Since $T \perp N$: $0 = \langle T, N \rangle \Rightarrow 0 = \frac{1}{v}\frac{d}{dt}\langle T, N \rangle$

$$= \langle \frac{1}{v}\dot{T}, N \rangle + \langle T, \frac{1}{v}\dot{N} \rangle$$

$\overset{ⓐ}{\Rightarrow}$ $\langle T, \frac{1}{v}\dot{N} \rangle = -\kappa$ ⓒ

If we express $\frac{1}{v}\dot{N} = aT + bN + cB$ then, since $\{T, N, B\}$ is orthonormal:

$a = \langle \frac{1}{v}\dot{N}, T \rangle = -\kappa$

$c = \langle \frac{1}{v}\dot{N}, B \rangle = \tau$

and $b = \langle \frac{1}{v}\dot{N}, N \rangle = 0$ since $\langle N, N \rangle = \text{const} = 1$

So $\frac{1}{v}\dot{N} = -\kappa T + \tau B$ ②

And: $B = T \times N$ so $\dot{B} = \dot{T} \times N + T \times \dot{N} \overset{①,②}{=} \kappa \cdot \underbrace{\cancel{N \times N}}_{0} + T \times (-\kappa T + \tau B)$

$$= \tau\, T \times B = -\tau N \quad ③$$

①, ②, ③ are the required relations.

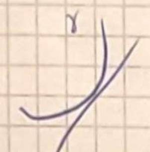
$\square$

The Frenet frame: Will be the natural kinematic frame following the evolution of γ.

- Sometimes: It is easier to use these relations to compute κ, τ.

What is the "geometric meaning" of κ, τ:

- κ: The rate at which γ leaves the tangent line



- τ: The "rate" at which γ leaves the osculating plane

<u>Proposition</u>: Let $\gamma: I \to \mathbb{R}^3$ be Frenet regular. γ is a planar curve (i.e. contained in a 2-plane) iff $\tau = 0$.

<u>Proof</u>: "$\Leftarrow$": If γ is contained in a plane Π, then this is the plane containing $\dot{\gamma}, \ddot{\gamma}$ $\Rightarrow$ $\Pi = \text{span}\{T_\gamma, N_\gamma\}$

So B is the unit normal to Π $\Rightarrow$ $B_\gamma(t) = \text{const}$.

$\Rightarrow \dot{B}_\gamma = 0$. By the Serret-Frenet formulas: $\tau_\gamma = 0$

"$\Rightarrow$":

If $\tau_\gamma \equiv 0$, By the Serret-Frenet formulas: $B_\gamma(t) = \text{const} = B$

Let $t_0 \in I$ and consider the plane Π passing through $\gamma(t_0)$ and with normal B

$$\Pi = \{x \in \mathbb{R}^3 : \langle x - \gamma(t_0), B \rangle = 0\}.$$

Consider the function $h(t) = \langle \gamma(t) - \gamma(t_0), B \rangle$. If we show that $h(t) = 0 \ \forall t$, then $\gamma \subseteq \Pi$.

We have: $h(t_0) = 0$ and, $\forall t \in I$:

$$\dot{h}(t) = \langle \dot{\gamma}(t), B \rangle = v_\gamma \cdot \langle T, B \rangle = 0$$

So $h(t) = \text{const} \Rightarrow h(t) = 0$. ☺

<u>Note</u>: In the same way that, for a planar curve, κ_γ measures the rate at which T_γ rotates (with respect to the natural parametrization), for a curve in $\mathbb{R}^3$: τ_γ measures the rate at which B_γ rotates around T_γ.

<u>Fundamental theorem of curves in $\mathbb{R}^3$:</u>

<u>Theorem</u>. Let $\kappa, \tau : I \to \mathbb{R}$ be <u>continuous</u> functions with $\kappa > 0$, there exists a curve $\gamma : I \to \mathbb{R}^3$ which is <u>Frenet regular</u>, such that (and naturally parametrized)

$$\kappa_\gamma(s) = \kappa(s), \quad \tau_\gamma(s) = \tau(s).$$

This curve is unique up to a <u>rigid motion</u> (i.e. an isometry of $\mathbb{R}^3$).

<u>Proof</u>: We will rely on the observation that, for

a Frenet regular curve γ, if we define the matrix

$$F_\gamma(t) = [\, T_\gamma(t) \,\vdots\, N_\gamma(t) \,\vdots\, B_\gamma(t) \,]$$

(the columns are the corresponding vectors) then, since $\{T_\gamma, N_\gamma, B_\gamma\}$ is a positively oriented orthonormal basis, we have $F_\gamma \in SO(3)$.

- In fact: F_γ is the rotation matrix that maps the usual Cartesian basis $\{e_1, e_2, e_3\}$ to $\{T, N, B\}$.

Then, the Serret-Frenet formulas take the following form when γ is naturally parametrized (i.e. $V_\gamma = 1$):

$$\frac{d}{ds} F_\gamma(s) = F_\gamma(s) \cdot \Omega_\gamma(s) \quad ①, \quad \text{where} \quad \Omega_\gamma(s) = \begin{pmatrix} 0 & k_\gamma(s) & 0 \\ -k_\gamma(s) & 0 & \tau_\gamma(s) \\ 0 & -\tau_\gamma(s) & 0 \end{pmatrix}$$

<u>Proof of uniqueness:</u>

Suppose that we have two such curves γ_1, γ_2 solving ①

with $\Omega(s) = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$.

By applying an isometry of $\mathbb{R}^3$ to either curve, I can assume without loss of generality that

1) $\gamma_1(0) = \gamma_2(0) = 0$

2) $\{T_{\gamma_1}(0), N_{\gamma_1}(0), B_{\gamma_1}(0)\} = \{T_{\gamma_2}(0), N_{\gamma_2}(0), B_{\gamma_2}(0)\} = \{e_1, e_2, e_3\}$

(So that $F_1(0) = F_2(0) = \mathbb{I}$).

Then both $F_1(s), F_2(s)$ solve the same equation ① (same $\Omega(s)$) with the same initial condition. We will show that this means that $F_1(s) = F_2(s)$:

$$\frac{d}{ds}(F_1 \cdot F_2^{-1}) \overset{F_2 \in SO(3)}{=} \frac{d}{ds}(F_1 \cdot F_2^T) = \dot{F}_1 \cdot F_2^T + F_1 \cdot \dot{F}_2^T$$

$$\overset{\text{equation ①}}{=} F_1 \cdot \Omega \cdot F_2^T + F_1 \cdot \Omega^T F_2^T$$

$$\overset{\Omega^T = -\Omega}{=} 0$$

So $F_1(s) F_2^{-1}(s) = F_1(0) F_2^{-1}(0) = \mathbb{I} \Rightarrow F_1(s) = F_2(s) \;\; \forall s$

In particular: $T_{\gamma_1}(s) = T_{\gamma_2}(s) \Leftrightarrow \dot{\gamma}_1(s) = \dot{\gamma}_2(s)$

Integrate in s: $\gamma_1(s) - \cancelto{0}{\gamma_1(0)} = \gamma_2(s) - \cancelto{0}{\gamma_2(0)} \;\; \forall s$.

Proof of existence:

Consider the matrix differential equation ①:

$$\begin{cases} \frac{d}{ds} F(s) = F(s) \cdot \Omega(s) \\ F(0) = \mathbb{I} \end{cases}$$

From Cauchy-Lipschitz theorem $\exists$ C^1 solution $F: I \longrightarrow M_3(\mathbb{R})$ (since $\Omega \in C^0$). Let's show that $F(s) \in SO(3)$:

$$\frac{d}{ds}(F \cdot F^T) = \dot{F} \cdot F^T + F \cdot \dot{F}^T = F \Omega F^T + F \Omega^T F^T \overset{\Omega = -\Omega^T}{=} 0$$

So $F(s) F^T(s) = F(0) \cdot F^T(0) = \mathbb{I}$ so $F \in SO(3)$

And $\det F(s) = \pm 1 \implies$ Since $\det F(s)$ is a continuous function: $\det F(s) = +1$ (it is $+1$ at $s=0$)

Let's call $T(s)$, $N(s)$, $B(s)$ the column of F.

Define the curve γ by

$$\gamma(s) = \int_0^s T(u)\, du.$$

Then $\dot{\gamma}(s) = T(s)$ (so T is the unit tangent and $\|\dot{\gamma}\| = 1$)

Then equation ① implies that

~~$N(s) \in \text{span}\{T(s), \dot{T}(s)\} = \text{span}\{$~~

$N(s) \parallel \dot{T}(s) = \ddot{\gamma}(s)$ so $N(s)$ is the principal normal.

So $B(s)$ is the binormal. (since $B_\gamma(s) = T(s) \times N(s) = B(s)$).

And equation ① is just the Serret-Frenet formula, so

$\kappa_\gamma(s) = \kappa(s)$, $\tau_\gamma(s) = \tau(s)$.